

The Fractional Chromatic Number of Triangle-free Graphs with $\Delta \leq 3$

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Abstract

Let G be a triangle-free graph with maximum degree at most 3. Staton proved that the independence number of G is at least $\frac{5}{14}|V(G)|$. Heckman and Thomas conjectured that Staton's result can be strengthened into a bound on the fractional chromatic number of G , namely $\chi_f(G) \leq \frac{14}{5}$. Recently, Hatami and Zhu proved that $\chi_f(G) \leq 3 - \frac{3}{64}$. In this paper, we prove $\chi_f(G) \leq 3 - \frac{3}{43}$.

1 Introduction

This paper investigates the fractional chromatic number of a triangle-free graph with maximum degree at most 3. For a simple (finite) graph G , the fractional chromatic number of G is the linear programming relaxation of the chromatic number of G . Let $\mathcal{I}(G)$ be the family of independent sets of G . A mapping $f: \mathcal{I}(G) \rightarrow [0, 1]$ is called an r -fractional coloring of G if $\sum_{S \in \mathcal{I}(G)} f(S) \leq r$ and $\sum_{v \in S, S \in \mathcal{I}(G)} f(S) \geq 1$ for each $v \in V(G)$. The *fractional chromatic number* $\chi_f(G)$ of G is the least r for which G has an r -fractional coloring.

Alternatively, the fractional chromatic number can also be defined through multiple colorings. A b -fold coloring of G assigns a set of b colors to each vertex such that any two adjacent vertices receive disjoint sets of colors. We say a graph G is $a:b$ -colorable if there is a b -fold coloring of G in which each color is drawn from a palette of a colors. We refer to such a coloring as an $a:b$ -coloring. The b -fold coloring number, denoted as $\chi_b(G)$, is the smallest integer a such that G has a $a:b$ -coloring. Note that $\chi_1(G) = \chi(G)$. It is known [13] that $\chi_b(G)$ (as a function of b) is sub-additive and so the $\lim_{b \rightarrow \infty} \frac{\chi_b(G)}{b}$ always exists, which turns out to be an alternative definition of $\chi_f(G)$. (Moreover, $\chi_f(G)$ is a rational number and the limit can be replaced by minimum.)

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Let $\chi(G)$ be the chromatic number of G and $\omega(G)$ be the clique number of G . We have the following simple relation,

$$\omega(G) \leq \chi_f(G) \leq \chi(G). \quad (1)$$

Now we consider a graph G with maximum degree $\Delta(G)$ at most three. If G is not K_4 , then G is 3-colorable by Brooks' theorem. If G contains a triangle, then $\chi_f(G) \geq \omega(G) = 3$. Equation (1) implies $\chi_f(G) = 3$. One may ask what is the possible value of $\chi_f(G)$ if G is triangle-free and $\Delta(G)$ is at most 3; this problem is motivated by a well-known and solved problem of determining the maximum independence number $\alpha(G)$ for such graphs. Staton [14] showed that

$$\alpha(G) \geq 5n/14 \quad (2)$$

for any triangle-free graph G on n vertices with maximum degree at most 3. Actually, Staton's bound is the best possible since the generalized Petersen graph $P(7, 2)$ has 14 vertices and independence number 5 as noticed by Fajtlowicz [3]. Griggs and Murphy [6] designed a linear-time algorithm to find an independent set in G of size at least $5(n - k)/14$, where k is the number of components of G that are 3-regular. Heckman and Thomas [8] gave a simpler proof of Staton's bound and designed a linear-time algorithm to find an independent set in G with size at least $5n/14$.

In the same paper [8], Heckman and Thomas conjectured

$$\chi_f(G) \leq \frac{14}{5} \quad (3)$$

for every triangle-free graph with maximum degree at most 3. Note that [13]

$$\chi_f(G) = \frac{n}{\alpha(G)}, \quad (4)$$

provided G is vertex transitive. Equation (4) implies that the generalized Petersen graph $P(7, 2)$ has the fractional chromatic number $\frac{14}{5}$. Thus, the conjecture is tight if it holds.

Recently, Hatami and Zhu [7] proved that $\chi_f(G) \leq 3 - \frac{3}{64}$, provided G is triangle-free with maximum degree at most three. Their idea is quite clever. For some independent set X , the graph obtained by identifying all neighbors of X into one fat vertex is 3-colorable. Now each vertex in X has freedom of choosing two colors. This observation results the improvement of $\chi_f(G)$. Here we improve their result and get the following theorem.

Theorem 1 *If G is triangle-free and has maximum degree at most 3, then $\chi_f(G) \leq 3 - \frac{3}{43}$.*

Comparing to Hatami and Zhu's result, here we only shrink the gap (to the conjectured value 2.8) by 15%. However, it is quite hard to obtain this improvement. The main idea is to extend the independent set X of G^* in Lemma 12 of [5] to the admissible set $X = X_1 \cup X_2 \cup X_3$, where X_1 , X_2 , and X_3

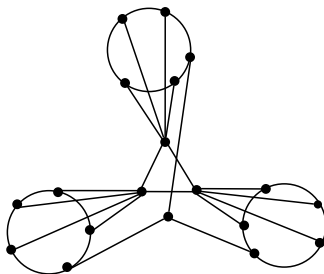


Figure 1: A difficult case of $G'(X)$: here the triangle in the middle is obtained from contracting the neighborhood of X_i in G , for $1 \leq i \leq 3$; the remaining part forms a Gallai tree. Note $\chi(G'(X)) = 4$.

are three independent sets in G^* . However, this extension causes fundamental difficulty (see the proof of Lemma 9); the reason is that in general $G'(X)$ (in the proof of Lemma 9) could be 4-chromatic as shown in Figure 1.

To get over the difficulty, we develop a heavy machinery of fractionally-critical graphs and use it to prove Lemma 7 and 8. The contribution of this paper is not just an improvement on $\chi_f(G)$ for triangle-free graph G with $\Delta(G) \leq 3$. More importantly, the theory developed in section 2 can be applied to more general scenarios concerning the fractional chromatic numbers of graphs. Using the tools developed in this paper, King-Lu-Peng [10] classified all connected graphs with $\chi_f(G) \geq \Delta(G)$. They [10] further proved that $\chi_f(G) \leq \Delta(G) - \frac{2}{67}$ for all graphs G such that G is K_Δ -free, $\Delta(G) \geq 3$, and G is neither C_8^2 (the square of C_8) nor $C_5 \boxtimes K_2$ (the strong product of C_5 and K_2) as shown in Figure 2. Very recently, Edwards and King [2] improve the lower bound on $\Delta(G) - \chi_f(G)$ for all $\Delta \geq 6$. The Heckman-Thomas' conjecture is a special case at $\Delta(G) = 3$. We also notice that a better bound $32/11$ (toward Heckman-Thomas' conjecture) was proved in [5] very recently.

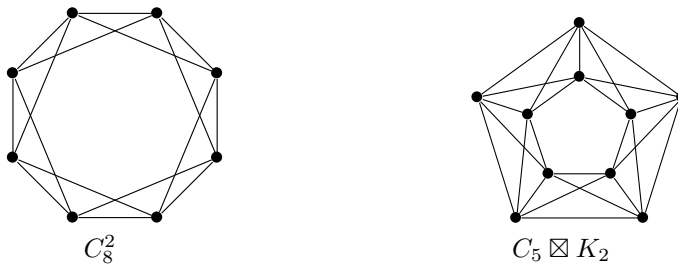


Figure 2: Two exception graphs with $\chi_f(G) = \Delta(G)$: C_8^2 and $C_5 \boxtimes K_2$.

The rest of the paper is organized as follows. In section 2, we will study the convex structure of fractional colorings and the fractionally-critical graphs. In section 3, we will prove several key lemmas. In last section, we will show that G can be partitioned into 42 admissible sets and present the proof of the main

theorem.

2 Lemmas and Notations

In this section, we introduce an alternative definition of “fractional colorings.” The new definition highlights the convex structure of the set of all fractional colorings. The extreme points of these “fractional colorings” play a central role in our proofs and seem to have independent interest. Our approach is analogous to defining rational numbers from integers.

2.1 Convex structures of fractional colorings

In this paper, we use bold letter \mathbf{c} to represent a coloring. Recall that a b -fold coloring of a graph G assigns a set of b colors to each vertex such that any two adjacent vertices receive disjoint sets of colors. Given a b -fold coloring \mathbf{c} , let $A(\mathbf{c}) = \cup_{v \in V(G)} \mathbf{c}(v)$ be the set of all colors used in \mathbf{c} . Two b -fold colorings \mathbf{c}_1 and \mathbf{c}_2 are *isomorphic* if there is a bijection $\phi: A(\mathbf{c}_1) \rightarrow A(\mathbf{c}_2)$ such that $\phi \circ \mathbf{c}_1 = \mathbf{c}_2$. In this case we write $\mathbf{c}_1 \cong \mathbf{c}_2$. The isomorphic relation \cong is an equivalence relation. We use $\bar{\mathbf{c}}$ to denote the isomorphic class in which \mathbf{c} belongs to. Whenever clear under the context, we will not distinguish a b -fold coloring \mathbf{c} and its isomorphic class $\bar{\mathbf{c}}$.

For a graph G and a positive integer b , let $\mathcal{C}_b(G)$ be the set of all (isomorphic classes of) b -fold colorings of G . For $\mathbf{c}_1 \in \mathcal{C}_{b_1}(G)$ and $\mathbf{c}_2 \in \mathcal{C}_{b_2}(G)$, we can define $\mathbf{c}_1 + \mathbf{c}_2 \in \mathcal{C}_{b_1+b_2}(G)$ as follows: for any $v \in V(G)$,

$$(\mathbf{c}_1 + \mathbf{c}_2)(v) = \mathbf{c}_1(v) \sqcup \mathbf{c}_2(v),$$

i.e., $\mathbf{c}_1 + \mathbf{c}_2$ assigns v the disjoint union of $\mathbf{c}_1(v)$ and $\mathbf{c}_2(v)$.

Let $\mathcal{C}(G) = \cup_{b=0}^{\infty} \mathcal{C}_b(G)$. It is easy to check that “+” is commutative and associative. Under the addition above, $\mathcal{C}(G)$ forms a commutative monoid with the unique 0-fold coloring (denoted by 0, for short) as the identity. For a positive integer t and $\mathbf{c} \in \mathcal{C}_b(G)$, we define

$$t \cdot \mathbf{c} = \overbrace{\mathbf{c} + \cdots + \mathbf{c}}^t$$

to be the new tb -fold coloring by duplicating each color t times.

For $\mathbf{c}_1 \in \mathcal{C}_{b_1}(G)$ and $\mathbf{c}_2 \in \mathcal{C}_{b_2}(G)$, we say \mathbf{c}_1 and \mathbf{c}_2 are *equivalent*, denoted as $\mathbf{c}_1 \sim \mathbf{c}_2$, if there exists a positive integer s such that $sb_2 \cdot \mathbf{c}_1 \cong sb_1 \cdot \mathbf{c}_2$. (This is an analogue of the classical definition of rational numbers with a slight modification. The multiplication by s is needed here because the division of a fractional coloring by an integer has not been defined yet. The proofs of Lemma 1 and 2 are straightforward and are omitted here.)

Lemma 1 *The binary relation \sim is an equivalence relation over $\mathcal{C}(G)$.*

Let $\mathcal{F}(G) = \mathcal{C}(G)/\sim$ be the set of all equivalence classes. Each equivalence class is called a fractional coloring of G . For any $\mathbf{c} \in \mathcal{C}_b(G)$, the equivalence class of \mathbf{c} under \sim is denoted by $\pi(\mathbf{c}) = \frac{\mathbf{c}}{b}$.

Remark: The notation $\frac{\mathbf{c}}{b}$ makes sense only when \mathbf{c} is a b -fold coloring.

Given any rational number $\lambda = \frac{q}{p} \in [0, 1]$ (with two positive integers p and q) and two fractional colorings (two equivalence classes) $\frac{\mathbf{c}_1}{b_1}$ and $\frac{\mathbf{c}_2}{b_2}$, we define the linear combination as

$$\lambda \mathbf{c}_1 + (1 - \lambda) \mathbf{c}_2 = \frac{qb_2 \cdot \mathbf{c}_1 + (p - q)b_1 \cdot \mathbf{c}_2}{pb_1b_2}.$$

The following lemma shows the definition above is independent of the choices of \mathbf{c}_1 and \mathbf{c}_2 , and so $\lambda \mathbf{c}_1 + (1 - \lambda) \mathbf{c}_2$ is a fractional coloring depending only on λ , $\frac{\mathbf{c}_1}{b_1}$, and $\frac{\mathbf{c}_2}{b_2}$.

Lemma 2 For $i \in \{1, 2, 3, 4\}$, let $\mathbf{c}_i \in \mathcal{C}_{b_i}(G)$. Suppose that $\mathbf{c}_1 \sim \mathbf{c}_3$ and $\mathbf{c}_2 \sim \mathbf{c}_4$. For any non-negative integers $p, q, p',$ and q' satisfying $\frac{q}{p} = \frac{q'}{p'} \in [0, 1]$, we have $qb_2 \cdot \mathbf{c}_1 + (p - q)b_1 \cdot \mathbf{c}_2 \sim q'b_4 \cdot \mathbf{c}_3 + (p' - q')b_3 \cdot \mathbf{c}_4$.

Define a function $g_G: \mathcal{F}(G) \rightarrow \mathbb{Q}$ by $g_G(\frac{\mathbf{c}}{b}) = \frac{|A(\mathbf{c})|}{b}$. If the graph G is clear under the context, then we write it as $g(\frac{\mathbf{c}}{b})$ for short. It is easy to check that g does not depend on the choice of \mathbf{c} and so g is well-defined. For any $\tau > 0$, we define

$$\mathcal{F}_\tau(G) = \left\{ \frac{\mathbf{c}}{b} \in \mathcal{F}(G) \mid g\left(\frac{\mathbf{c}}{b}\right) \leq \tau \right\}.$$

A fractional coloring \mathbf{c} is called *extremal* in $\mathcal{F}_\tau(G)$ if it can not be written as a linear combination of two (or more) distinct fractional colorings in $\mathcal{F}_\tau(G)$.

Theorem 2 For any graph G on n vertices, there is an embedding $\phi: \mathcal{F}(G) \rightarrow \mathbb{Q}^{2^n-1}$ such that ϕ keeps convex structure. Moreover, for any rational number τ , $\mathcal{F}_\tau(G)$ is the convex hull of some extremal fractional colorings.

Proof: We would like to define $\phi: \mathcal{F}(G) \rightarrow \mathbb{Q}^{2^n-1}$ as follows.

Given a fractional coloring $\frac{\mathbf{c}}{b}$, we can fill these colors into the regions of the general Venn Diagram on n -sets. For $1 \leq i \leq 2^n - 1$, we can write i as a binary string $a_1a_2 \cdots a_n$ such that $a_v \in \{0, 1\}$ for all $1 \leq v \leq n$. Write $\mathbf{c}^1(v) = \mathbf{c}(v)$ and $\mathbf{c}^0(v) = \overline{\mathbf{c}(v)}$ (the complement set of $\mathbf{c}(v)$); then the number of colors in i -th region of the Venn Diagram can be written as

$$h_i(\mathbf{c}) = |\cap_{v=1}^n \mathbf{c}^{a_v}(v)|.$$

By the definition, h_i is additive, i.e.

$$h_i(\mathbf{c}_1 + \mathbf{c}_2) = h_i(\mathbf{c}_1) + h_i(\mathbf{c}_2).$$

Thus $\frac{h_i(\mathbf{c})}{b}$ depends only on the fractional coloring $\frac{\mathbf{c}}{b}$ but not on \mathbf{c} itself.

The i -th coordinate of $\phi(\frac{\mathbf{c}}{b})$ is defined to be

$$\phi_i\left(\frac{\mathbf{c}}{b}\right) = \frac{h_i(\mathbf{c})}{b}.$$

It is easy to check that ϕ_i is a well-defined function on $\mathcal{F}(G)$. Moreover, for any $\lambda = \frac{q}{p} \in [0, 1]$ and any two fractional colorings $\frac{\mathbf{c}_1}{b_1}$ and $\frac{\mathbf{c}_2}{b_2}$, we have

$$\begin{aligned}
\phi_i \left(\lambda \frac{\mathbf{c}_1}{b_1} + (1 - \lambda) \frac{\mathbf{c}_2}{b_2} \right) &= \phi_i \left(\frac{qb_2 \cdot \mathbf{c}_1 + (p - q)b_1 \cdot \mathbf{c}_2}{pb_1b_2} \right) \\
&= \frac{h_i(qb_2 \cdot \mathbf{c}_1 + (p - q)b_1 \cdot \mathbf{c}_2)}{pb_1b_2} \\
&= \frac{qb_2h_i(\mathbf{c}_1) + (p - q)b_1h_i(\mathbf{c}_2)}{pb_1b_2} \\
&= \frac{q}{p} \frac{h_i(\mathbf{c}_1)}{b_1} + \left(1 - \frac{q}{p}\right) \frac{h_i(\mathbf{c}_2)}{b_2} \\
&= \lambda \phi_i \left(\frac{\mathbf{c}_1}{b_1} \right) + (1 - \lambda) \phi_i \left(\frac{\mathbf{c}_2}{b_2} \right).
\end{aligned}$$

Thus ϕ keeps the convex structure.

It remains to show ϕ is a one-to-one mapping. Assume $\phi(\frac{\mathbf{c}_1}{b_1}) = \phi(\frac{\mathbf{c}_2}{b_2})$. We need to show $\mathbf{c}_1 \sim \mathbf{c}_2$. Let $\mathbf{c}'_1 = b_2 \cdot \mathbf{c}_1$ and $\mathbf{c}'_2 = b_1 \cdot \mathbf{c}_2$. Both \mathbf{c}'_1 and \mathbf{c}'_2 are b_1b_2 -fold colorings. Note $\phi(\frac{\mathbf{c}'_1}{b_1b_2}) = \phi(\frac{\mathbf{c}'_2}{b_1b_2})$. For $j \in \{1, 2\}$ and $1 \leq i \leq 2^n - 1$, we denote the set of colors in the i -th Venn Diagram region of $A(\mathbf{c}'_j)$ by $B_i(\mathbf{c}'_j)$. Since $\phi(\frac{\mathbf{c}'_1}{b_1b_2}) = \phi(\frac{\mathbf{c}'_2}{b_1b_2})$, we have

$$|B_i(\mathbf{c}'_1)| = |B_i(\mathbf{c}'_2)|$$

for $1 \leq i \leq 2^n - 1$. There is a bijection ψ_i from $B_i(\mathbf{c}'_1)$ to $B_i(\mathbf{c}'_2)$. Note that for $j \in \{1, 2\}$, we have a partition of $A(\mathbf{c}'_j)$:

$$A(\mathbf{c}'_j) = \sqcup_{i=1}^{2^n-1} B_i(\mathbf{c}'_j).$$

Define a bijection ψ from $A(\mathbf{c}'_1)$ to $A(\mathbf{c}'_2)$ be the union of all ψ_i ($1 \leq i \leq 2^n - 1$). We have

$$\psi \circ \mathbf{c}'_1 = \mathbf{c}'_2.$$

Thus $\mathbf{c}'_1 \cong \mathbf{c}'_2$. Note that $\mathbf{c}_1 \sim \mathbf{c}'_1$ and $\mathbf{c}_2 \sim \mathbf{c}'_2$. We conclude that $\mathbf{c}_1 \sim \mathbf{c}_2$.

Under the embedding, $\phi(\mathcal{F}_\tau(G))$ consists of all rational points in a polytope defined by the intersection of finite number of half spaces. Note that all coefficients of the equations of hyperplanes are rational. Each rational point in the polytope corresponds to a fractional coloring while each vertex of the polytope corresponds to an extremal fractional coloring. \square

Remark: It is well-known that for any graph G there is a $a:b$ -coloring of G with $\frac{a}{b} = \chi_f(G)$. In our terminology, we have

$$\chi_f(G) = \min \left\{ g\left(\frac{\mathbf{c}}{b}\right) \mid \text{for any } \frac{\mathbf{c}}{b} \in \mathcal{F}(G) \right\}.$$

2.2 Coloring restriction and extension

Let H be a subgraph of G . A b -fold coloring of G is naturally a b -fold coloring of H ; this *restriction* operation induces a mapping $i_G^H: \mathcal{F}(G) \rightarrow \mathcal{F}(H)$. It is

easy to check that i_G^H keeps convex structure, i.e., for any $\frac{c_1}{b_1}, \frac{c_2}{b_2} \in \mathcal{F}(G)$ and $\lambda \in [0, 1] \cap \mathbb{Q}$, we have

$$i_G^H \left(\lambda \frac{c_1}{b_1} + (1 - \lambda) \frac{c_2}{b_2} \right) = \lambda i_G^H \left(\frac{c_1}{b_1} \right) + (1 - \lambda) i_G^H \left(\frac{c_2}{b_2} \right).$$

It is also trivial that

$$g_H \left(i_G^H \left(\frac{c}{b} \right) \right) \leq g_G \left(\frac{c}{b} \right).$$

Now we consider a reverse operation. We say a fractional coloring $\frac{c_1}{b_1} \in \mathcal{F}(H)$ is *extensible* in $\mathcal{F}_t(G)$ if there is a fractional coloring $\frac{c}{b} \in \mathcal{F}_t(G)$ satisfying

$$i_G^H \left(\frac{c}{b} \right) = \frac{c_1}{b_1}.$$

We say a fractional coloring $\frac{c_1}{b_1} \in \mathcal{F}(H)$ is *fully extensible* in $\mathcal{F}(G)$ if it is extensible in $\mathcal{F}_t(G)$, where $t = g_H(\frac{c_1}{b_1})$. (It also implies that $\frac{c_1}{b_1}$ is extensible in $\mathcal{F}_t(G)$ for all $t \geq g_H(\frac{c_1}{b_1})$.)

Lemma 3 *Let H be a subgraph of G and $\frac{c_i}{b_i} \in \mathcal{F}(H)$ for $i \in \{1, 2\}$. Assume that for $i \in \{1, 2\}$, $\frac{c_i}{b_i}$ is fully extensible in $\mathcal{F}(G)$. For any $\lambda \in \mathbb{Q} \cap [0, 1]$, we have $\lambda \frac{c_1}{b_1} + (1 - \lambda) \frac{c_2}{b_2}$ is fully extensible in $\mathcal{F}(G)$.*

Proof: Let $t_i = g_H(\frac{c_i}{b_i})$ for $i \in \{1, 2\}$. Note that there are fractional colorings $\frac{c'_i}{b'_i} \in \mathcal{F}_{t_i}(G)$ such that $i_G^H \left(\frac{c'_i}{b'_i} \right) = \frac{c_i}{b_i}$ for $i \in \{1, 2\}$. Let $\frac{c}{b} = \lambda \frac{c'_1}{b'_1} + (1 - \lambda) \frac{c'_2}{b'_2}$. We have $g_G(\frac{c}{b}) = \lambda t_1 + (1 - \lambda) t_2$ and,

$$i_G^H \left(\frac{c}{b} \right) = \lambda i_G^H \left(\frac{c'_1}{b'_1} \right) + (1 - \lambda) i_G^H \left(\frac{c'_2}{b'_2} \right) = \lambda \frac{c_1}{b_1} + (1 - \lambda) \frac{c_2}{b_2}.$$

Note that

$$g_H \left(\lambda \frac{c_1}{b_1} + (1 - \lambda) \frac{c_2}{b_2} \right) = \lambda t_1 + (1 - \lambda) t_2.$$

Therefore, $\lambda \frac{c_1}{b_1} + (1 - \lambda) \frac{c_2}{b_2}$ is fully extensible in $\mathcal{F}(G)$ by the definition. \square

We say $G = G_1 \cup G_2$ if $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. Similarly, we say $H = G_1 \cap G_2$ if $V(H) = V(G_1) \cap V(G_2)$ and $E(H) = E(G_1) \cap E(G_2)$.

Lemma 4 *Let G be a graph. Assume that G_1 and G_2 are two subgraphs such that $G_1 \cup G_2 = G$ and $G_1 \cap G_2 = H$. If two fractional colorings $\frac{c_1}{b_1} \in \mathcal{F}(G_1)$ and $\frac{c_2}{b_2} \in \mathcal{F}(G_2)$ satisfy $i_{G_1}^H(\frac{c_1}{b_1}) = i_{G_2}^H(\frac{c_2}{b_2})$, then there exists a fractional coloring $\frac{c}{b} \in \mathcal{F}(G)$ satisfying*

$$i_G^{G_i} \left(\frac{c}{b} \right) = \frac{c_i}{b_i}$$

for $i \in \{1, 2\}$ and $g_G(\frac{c}{b}) = \max\{g_{G_1}(\frac{c_1}{b_1}), g_{G_2}(\frac{c_2}{b_2})\}$.

Proof: Without loss of generality, we can assume $b_1 = b_2 = b$ (by taking the least common multiple if it is necessary.) We also assume $g_{G_1}(\frac{\mathbf{c}_1}{b_1}) \leq g_{G_2}(\frac{\mathbf{c}_2}{b_2})$, then we have $|A_{G_1}(\mathbf{c}_1)| \leq |A_{G_2}(\mathbf{c}_2)|$. Since $i_{G_1}^H(\frac{\mathbf{c}_1}{b_1}) = i_{G_2}^H(\frac{\mathbf{c}_2}{b_2})$, there is a bijection ϕ from $\cup_{v \in V(H)} \mathbf{c}_1(v)$ to $\cup_{v \in V(H)} \mathbf{c}_2(v)$. Extend ϕ as an one-to-one mapping from $A_{G_1}(\mathbf{c}_1)$ to $A_{G_2}(\mathbf{c}_2)$ in an arbitrary way. Now we define a b -fold coloring \mathbf{c} of G as follows.

$$\mathbf{c}(v) = \begin{cases} \phi(\mathbf{c}_1(v)) & \text{if } v \in V(G_1), \\ \mathbf{c}_2(v) & \text{if } v \in V(G_2). \end{cases}$$

Since G_1 and G_2 cover all edges of G , \mathbf{c} is well-defined. Note $\mathbf{c}|_{V(G_1)} = \phi \circ \mathbf{c}_1 \cong \mathbf{c}_1$ and $\mathbf{c}|_{V(G_2)} = \mathbf{c}_2$. Thus for $i \in \{1, 2\}$, we have

$$i_G^{G_i}(\frac{\mathbf{c}}{b}) = \frac{\mathbf{c}_i}{b_i}.$$

We also have $g_G(\frac{\mathbf{c}}{b}) = \frac{|A_{G_2}(\mathbf{c}_2)|}{b} = g_{G_2}(\frac{\mathbf{c}_2}{b_2})$. \square

Theorem 3 *Let G be a graph. Assume that G_1 and G_2 are two subgraphs such that $G_1 \cup G_2 = G$ and $G_1 \cap G_2 = H$. If $\chi_f(G_1) \leq t$ and any extreme fractional coloring in $\mathcal{F}_t(H)$ is extensible in $\mathcal{F}_t(G_2)$, then we have $\chi_f(G) \leq t$.*

Proof: There is a fractional coloring $\frac{\mathbf{c}}{b} \in \mathcal{F}(G_1)$ satisfying $g_{G_1}(\frac{\mathbf{c}}{b}) = t$. Since every extreme fractional coloring of $\mathcal{F}_t(H)$ is extensible in $\mathcal{F}_t(G_2)$, there exist fractional colorings $\frac{\mathbf{c}_1}{b_1}, \frac{\mathbf{c}_2}{b_2}, \dots, \frac{\mathbf{c}_r}{b_r} \in \mathcal{F}_t(G_2)$ such that $i_{G_2}^H(\frac{\mathbf{c}_1}{b_1}), i_{G_2}^H(\frac{\mathbf{c}_2}{b_2}), \dots, i_{G_2}^H(\frac{\mathbf{c}_r}{b_r})$ are all extreme fractional colorings in $\mathcal{F}_t(H)$. The fractional coloring $i_{G_1}^H(\frac{\mathbf{c}}{b})$ can be written as a linear combination of the extreme ones. Hence, there exist $\lambda_1, \lambda_2, \dots, \lambda_r \in \mathbb{Q} \cap [0, 1]$ such that $\sum_{i=1}^r \lambda_i = 1$ and

$$i_{G_1}^H(\frac{\mathbf{c}}{b}) = \sum_{i=1}^r \lambda_i i_{G_2}^H(\frac{\mathbf{c}_i}{b_i}).$$

Let $\frac{\mathbf{c}'}{b'} = \sum_{i=1}^r \lambda_i \frac{\mathbf{c}_i}{b_i} \in \mathcal{F}_t(G_2)$. We have

$$i_{G_1}^H(\frac{\mathbf{c}}{b}) = i_{G_2}^H(\frac{\mathbf{c}'}{b'}).$$

Applying Lemma 4, there exists a fractional coloring $\frac{\mathbf{c}''}{b''} \in \mathcal{F}_t(G)$. Therefore, we have $\chi_f(G) \leq t$. \square

Corollary 1 *Let G be a graph. Assume that G_1 and G_2 are two subgraphs such that $G_1 \cup G_2 = G$ and $G_1 \cap G_2 = K_r$ for some positive integer r . We have*

$$\chi_f(G) = \max\{\chi_f(G_1), \chi_f(G_2)\}.$$

Proof: Without loss of generality, we assume $\chi_f(G_1) \geq \chi_f(G_2)$. Let $t = \chi_f(G_1)$. We have $t \geq r$ as G_1 contains K_r . Since $\chi_f(G_2) \leq t$, we have $\mathcal{F}_t(G_2) \neq \emptyset$. Since H is a complete graph, $\mathcal{F}_t(H)$ contains only one fractional coloring;

namely, color all vertices of H using distinct colors. It is trivial that any extreme fractional coloring in $\mathcal{F}_t(H)$ is extensible in $\mathcal{F}_t((G_2))$. Applying Theorem 3, we have $\chi_f(G) \leq t$. The other direction is trivial. \square

Let uv be a non-edge of a graph G_2 . We denote $G_2 + uv$ to be the supergraph of G_2 by adding the edge uv and denote G_2/uv be the quotient graph by identifying the vertex u and the vertex v .

Lemma 5 *Let G be a graph. Assume that G_1 and G_2 are two subgraphs such that $G_1 \cup G_2 = G$ and $V(G_1) \cap V(G_2) = \{u, v\}$.*

1. *If uv is an edge of G , then we have*

$$\chi_f(G) = \max\{\chi_f(G_1), \chi_f(G_2)\}.$$

2. *If uv is not an edge of G , then we have*

$$\chi_f(G) \leq \max\{\chi_f(G_1), \chi_f(G_2 + uv), \chi_f(G_2/uv)\}.$$

Proof: Part 1 is a simple application of Corollary 1. For the proof of part 2, let $t = \max\{\chi_f(G_1), \chi_f(G_2 + uv), \chi_f(G_2/uv)\}$. Note $t \geq 2$. Let E_{uv} be the empty graph on the set of two vertices u and v . All fractional colorings of $\mathcal{F}_t(E_{uv})$ can be represented by the following weighted Venn Diagram, see Figure 3.

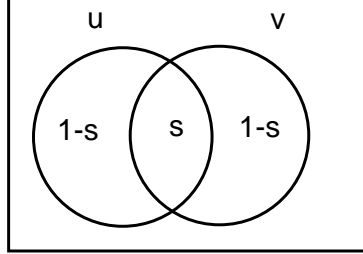


Figure 3: Fractional colorings on the empty graph on two vertices u and v are represented as a weighted Venn Diagram.

The parameter s measure the fraction of common colors shared by u and v . There are two extreme points in the convex hull: $s = 0$ and $s = 1$. The extremal fractional coloring corresponding to $s = 0$ is extensible in $\mathcal{F}_t(G_2)$ since $\chi_f(G + uv) \leq t$. The extremal fractional coloring corresponding to $s = 1$ is extensible in $\mathcal{F}_t(G_2)$ since $\chi_f(G/uv) \leq t$. Applying Theorem 3 and Lemma 3, we have $\chi_f(G) \leq t$. Part 2 is proved. \square

2.3 Fractionally-critical graphs

In this subsection, we will apply our machinery to triangle-free graphs with maximum degree at most 3.

Recall that a graph G is k -critical (for a positive integer k) if $\chi(G) = k$ and $\chi(H) < k$ for any proper subgraph H of G . For any rational number $t \geq 2$, a graph G is t -fractionally-critical if $\chi_f(G) = t$ and $\chi_f(H) < t$ for any proper subgraph H of G . For simplicity, we say G is *fractionally-critical* if G is $\chi_f(G)$ -fractionally-critical.

We will study the properties of fractionally-critical graphs. The following lemma is a consequence of Corollary 1.

Lemma 6 *Assume that G is a fractionally-critical graph with $\chi_f(G) \geq 2$. We have G is 2-connected. Moreover, if G has a vertex-cut $\{u, v\}$, then uv is not an edge of G .*

For any vertex u of a graph G and a positive integer i , we define $N_G^i(u) = \{v \in V : v \neq u \text{ and there is a path of length } i \text{ connecting } u \text{ and } v\}$.

Lemma 7 *Assume that G is a fractionally-critical triangle-free graph satisfying $\Delta(G) \leq 3$ and $\frac{11}{4} < \chi_f(G) < 3$. For any vertex $x \in V(G)$ and any 5-cycle C of G , we have either $|V(C) \cap N_G^2(x)| \leq 3$ or $|V(C) \cap N_G^1(x)| \geq 1$.*

Proof: Let $t = \chi_f(G)$. We have $\frac{11}{4} < t < 3$. We will prove the statement by contradiction. Suppose that there is a vertex x and a 5-cycle C satisfying $|V(C) \cap N_G^2(x)| \geq 4$ and $V(C) \cap N_G^1(x) = \emptyset$. Combined with the fact G being triangle-free, we have the following two cases.

Case 1: $|V(C) \cap N_G^2(x)| = 5$. Since $\Delta(G) \leq 3$ and G is triangle-free, it is easy to check that G contains the following subgraph G_9 as shown in Figure 4. Since G is 2-connected, G_9 is the entire graph (in [7]). Thus $\chi_f(G) \leq \frac{8}{3} < t$. Contradiction!

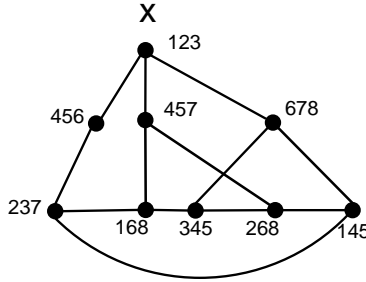


Figure 4: An 8:3-coloring of G_9 , where $|V(C) \cap N_G^2(x)| = 5$.

Case 2: $|V(C) \cap N_G^2(x)| = 4$ and there exists one vertex of C having distance of 3 to x . Hatami and Zhu [7] showed that G contains one of the five graphs in Figure 5 as a subgraph. (Note that some of the marked vertices u , v , and w may be missing or overlapped; these degenerated cases result in a smaller vertex-cut, and can be covered in a similar but easier way; we will discuss them at the end of this proof.)

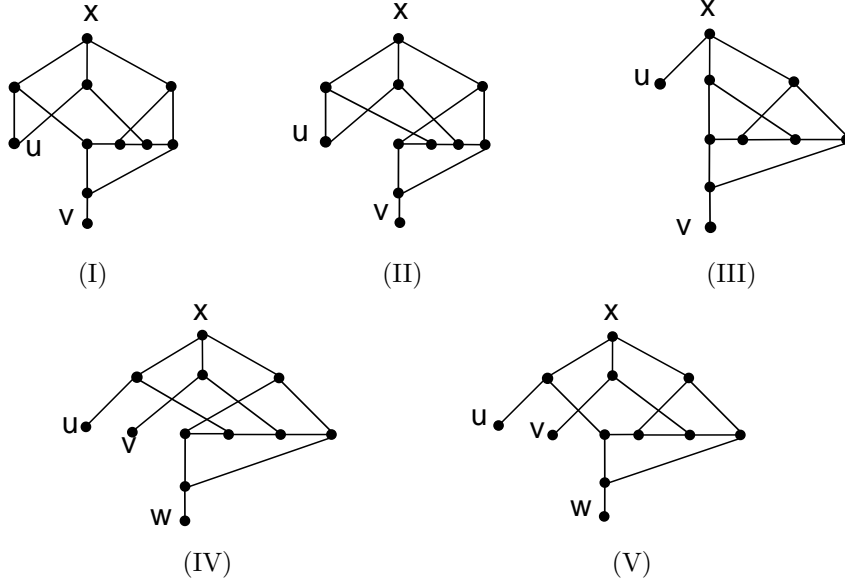


Figure 5: All possible cases of $|V(C) \cap N_G^2(x)| = 4$ and $|V(C) \setminus (N_G^1(x) \cup N_G^2(x))| = 1$.

If G contains a subgraph of type (I), (II), or (III), then G has a vertex-cut $\{u, v\}$. Let G_1 and G_2 be the two connected subgraphs of G such that $G_1 \cup G_2 = G$, $V(G_1) \cap V(G_2) = \{u, v\}$, and $x \in G_2$. In all three cases, $G_2 + uv$ and G_2/uv are 8:3-colorable. Please see Figure 6.

Applying Lemma 5, we have

$$\begin{aligned} \chi_f(G) &\leq \max\{\chi_f(G_1), \chi_f(G_2/uv), \chi_f(G_2 + uv)\} \\ &\leq \max\left\{\chi_f(G_1), \frac{8}{3}\right\}. \end{aligned}$$

Since $\chi_f(G) \geq t > \frac{8}{3}$, we must have $\chi_f(G_1) \geq \chi_f(G) = t$, which is a contradiction to the assumption that G is fractionally-critical.

If G contains one of the subgraphs (IV) and (V), then G has a vertex-cut set $H = \{u, v, w\}$ as shown in Figure 5. Let G_1 and G_2 be the two connected subgraphs of G such that $G_1 \cup G_2 = G$, $G_1 \cap G_2 = \{u, v, w\}$, and $x \in G_2$. We shall show $\chi_f(G_1) = t = \chi_f(G)$. Suppose not. We can assume $\chi_f(G_1) \leq t_0 < t$, where $\frac{8}{3} < \frac{11}{4} < t_0 < t < 3$. By Theorem 2, every fractional coloring in $\mathcal{F}_{t_0}(H)$ can be represented by a rational point in a convex polytope $\phi(\mathcal{F}_{t_0}(H))$. From now on, we will not distinguish the rational point in the convex polytope and the fractional coloring. Note the convex polytope for $\mathcal{F}_{t_0}(H)$ can be parametrized

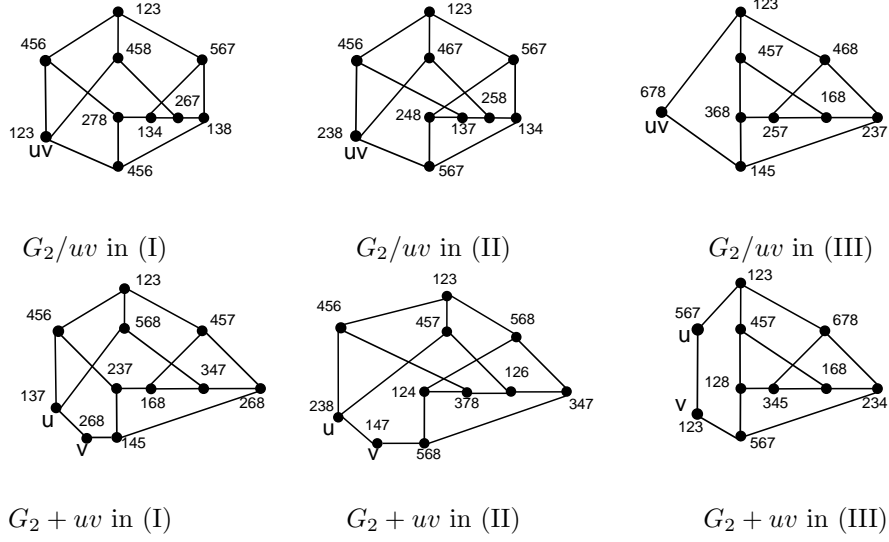


Figure 6: $G_2 + uv$ and G_2/uv are all 8:3-colorable in cases (I), (II), and (III).

as

$$\mathcal{F}_{t_0}(H) \cong \left\{ (x, y, z, s) \in \mathbb{Q}^4 \left| \begin{array}{l} x + y + s \leq 1 \\ x + z + s \leq 1 \\ y + z + s \leq 1 \\ 3 - x - y - z - 2s \leq t_0 \\ x, y, z, s \geq 0 \end{array} \right. \right\}.$$

See the weighted Venn Diagram in Figure 7.

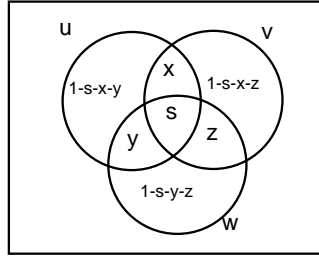


Figure 7: The general fractional colorings on the vertices u , v , and w .

The extreme fractional colorings of $\mathcal{F}_{t_0}(H)$ are represented (under ϕ , see Theorem 2) by:

- (a) $x = y = z = 0$ and $s = 1$.
- (b) $x = 1$ and $y = z = s = 0$.
- (c) $y = 1$ and $x = z = s = 0$.

- (d) $z = 1$ and $x = y = s = 0$.
- (e) $x = 3 - t_0$ and $y = z = s = 0$.
- (f) $y = 3 - t_0$ and $x = z = s = 0$.
- (g) $z = 3 - t_0$ and $x = y = s = 0$.

We will show that all 7 extreme fractional colorings are extensible in $\mathcal{F}_{t_0}(G_2)$.

- (a) Let G_2/uvw be the quotient graph by identifying u, v , and w as one vertex. The fractional coloring $(0, 0, 0, 1)$ is extensible in $\mathcal{F}_{t_0}(G_2)$ if and only if $\chi_f(G_2/uvw) \leq t_0$, which is verified by Figure 8.

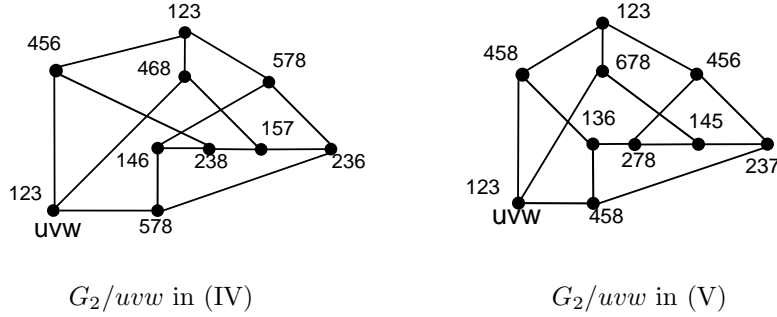


Figure 8: G_2/uvw are 8:3-colorable for subgraphs (IV) and (V).

- (b) Let $(G_2/vw) + u(vw)$ be the graph obtained by identifying v and w as one vertex vw followed by adding an edge $u(vw)$. The fractional coloring $(0, 0, 1, 0)$ is extensible in $\mathcal{F}_{t_0}(G_2)$ if and only if $\chi_f((G_2/vw) + u(vw)) \leq t_0$, which is verified by Figure 9.

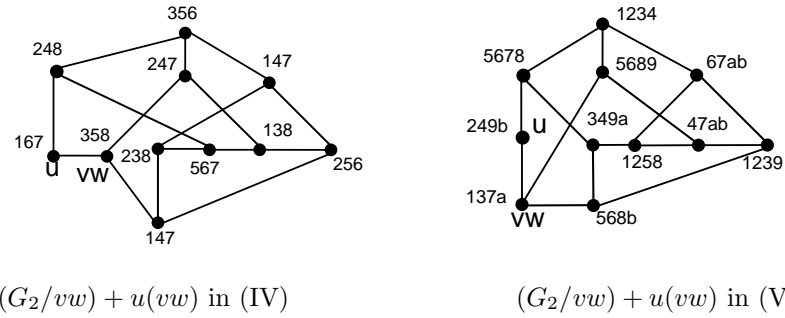
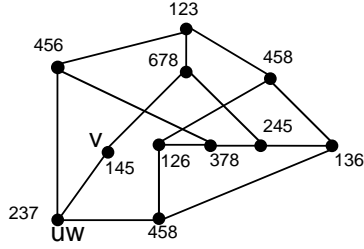
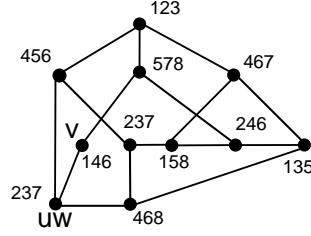


Figure 9: $(G_2/vw) + u(vw)$ in (IV) is 8:3-colorable, while $(G_2/vw) + u(vw)$ in (V) is 11:4-colorable.

- (c) Let $(G_2/uv) + v(uw)$ be the graph obtained by identifying u and w as one vertex uw followed by adding an edge $v(uw)$. The fractional coloring $(0, 1, 0, 0)$ is extensible in $\mathcal{F}_{t_0}(G_2)$ if and only if $\chi_f((G_2/uv) + v(uw)) \leq t_0$, which is verified by Figure 10.



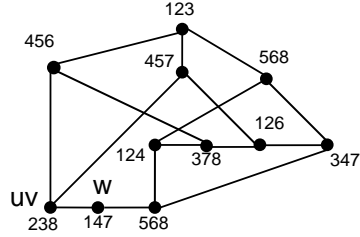
$(G_2/uv) + v(uw)$ in (IV)



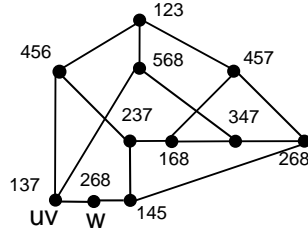
$(G_2/uv) + v(uw)$ in (V)

Figure 10: Both $(G_2/uv) + v(uw)$ in (IV) and (V) are 8:3-colorable.

- (d) Let $(G_2/uv) + w(uv)$ be the graph obtained by identifying u and v as one vertex uv followed by adding an edge $w(uv)$. The fractional coloring $(1, 0, 0, 0)$ is extensible in $\mathcal{F}_{t_0}(G_2)$ if and only if $\chi_f((G_2/uv) + w(uv)) \leq t_0$, which is verified by Figure 11.



$(G_2/uv) + w(uv)$ in (IV)



$(G_2/uv) + w(uv)$ in (V)

Figure 11: Both $(G_2/uv) + w(uv)$ in (IV) and (V) are 8:3-colorable.

- (e) Choose $\lambda = 3t_0 - 8$. Since $\frac{8}{3} < \frac{11}{4} < t_0 < 3$, we have $0 < \lambda < 1$. Note that

$$(3 - t_0, 0, 0, 0) = \lambda(0, 0, 0, 0) + (1 - \lambda)(\frac{1}{3}, 0, 0, 0).$$

Observe that $(3 - t_0, 0, 0, 0)$ being fully extensible in $\mathcal{F}(G_2)$ implies that $(3 - t_0, 0, 0, 0)$ is extensible in $\mathcal{F}_{t_0}(G_2)$. To show $(3 - t_0, 0, 0, 0)$ is fully extensible in $\mathcal{F}(G_2)$, it suffices to show both $(0, 0, 0, 0)$ and $(\frac{1}{3}, 0, 0, 0)$ are fully extensible in $\mathcal{F}(G_2)$ by Lemma 3 (see Figure 12 and 13).

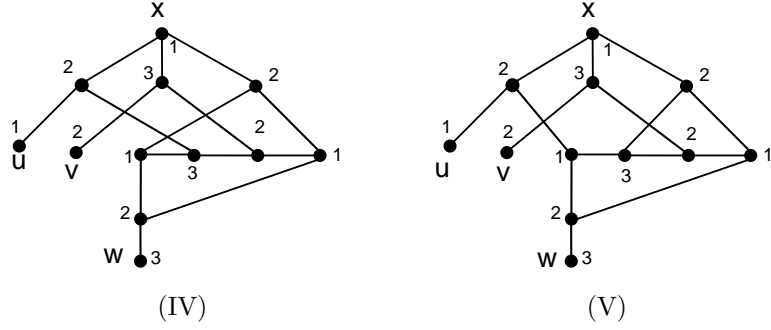


Figure 12: The fractional coloring $(0, 0, 0, 0)$ is fully extensible in $\mathcal{F}(G_2)$.

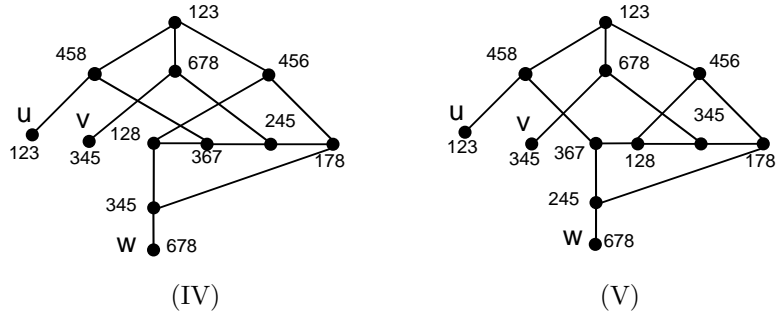


Figure 13: The fractional coloring $(\frac{1}{3}, 0, 0, 0)$ is fully extensible in $\mathcal{F}(G_2)$.

Contradiction! Hence, Lemma 7 follows. \square

Lemma 8 *Assume that G is a fractionally-critical triangle-free graph satisfying $\Delta(G) \leq 3$ and $\frac{8}{3} < \chi_f(G) < 3$. For any vertex $x \in V(G)$ and any 7-cycle C of G , we have $|V(C) \cap N_G^2(x)| \leq 5$.*

Proof: We prove the statement by contradiction. Suppose that there is a vertex x and a 7-cycle C satisfying $|V(C) \cap N_G^2(x)| \geq 6$. Recall that $|N_G^2(x)| \leq 6$. We have $|V(C) \cap N_G^2(x)| = 6$. Combined with the fact that G is triangle-free and 2-connected, G must be one of the following graphs in Figure 16. All of them are 8:3-colorable, see Figure 16. Contradiction!

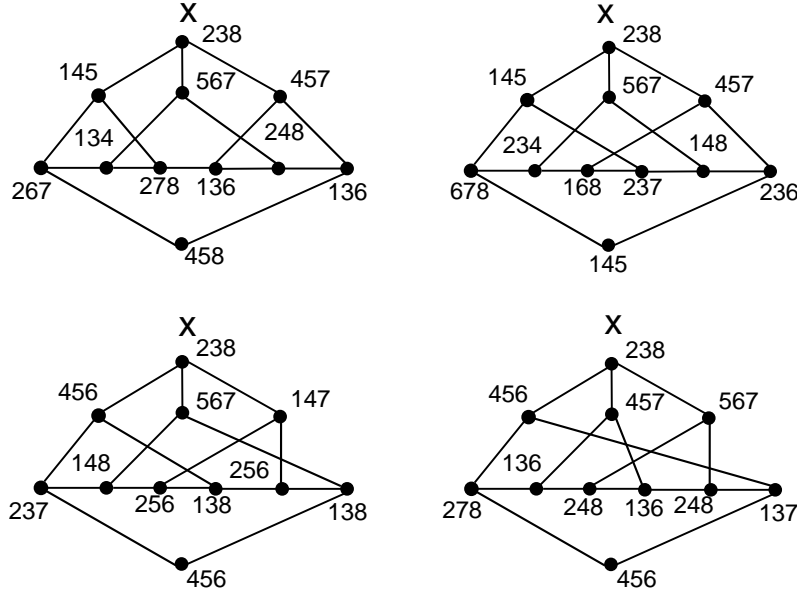


Figure 16: All possible graphs with $|V(C) \cap N_G^2(x)| = 6$.

\square

3 Admissible sets and Theorem 4

The following approach is similar to the one used in [7]. A significant difference is a new concept “admissible set”. Basically, it replaces the independent set X (of G^* in [7]) by three independent sets $X_1 \cup X_2 \cup X_3$.

Recall that $v \in N_G^i(u)$ if there exists a uv -path of length i in G . A set $X \subset V(G)$ is called *admissible* if X can be partitioned into three sets X_1, X_2, X_3 satisfying

1. If $\{u, v\} \subset X_i$ for some $i \in \{1, 2, 3\}$, then $v \notin N_G^1(u) \cup N_G^3(u) \cup N_G^5(u)$.

2. If $u \in X_i$ and $v \in X_j$ for some i and j satisfying $1 \leq i \neq j \leq 3$, then $v \notin N_G^1(u) \cup N_G^2(u) \cup N_G^4(u)$.

The following key theorem connects the $\chi_f(G)$ to a partition of G into admissible sets.

Theorem 4 *Assume that G is a fractionally-critical triangle-free graph satisfying $\Delta(G) \leq 3$ and $\frac{11}{4} < \chi_f(G) < 3$. If $V(G)$ can be partitioned into k admissible sets, then*

$$\chi_f(G) \leq 3 - \frac{3}{k+1}. \quad (5)$$

Let $X = X_1 \cup X_2 \cup X_3$ be an admissible set. Inspired by the method used in [7], we define an auxiliary graph $G' = G'(X)$ as follows. We use the notation $\Gamma(X)$ to denote the neighborhood of X in G . For each $i \in \{1, 2, 3\}$, let $Y_i = \Gamma(X_i)$. By admissible conditions, Y_1 , Y_2 , and Y_3 are all independent sets of G . Let G' be a graph obtained from G by deleting X ; identifying each Y_i as a single vertex y_i for $1 \leq i \leq 3$; and adding three edges y_1y_2 , y_2y_3 , y_1y_3 . We have the following lemma, which will be proved later.

Lemma 9 *Assume that G is fractionally-critical triangle-free graph satisfying $\Delta(G) \leq 3$ and $\frac{11}{4} < \chi_f(G) < 3$. Let X be an admissible set of G and $G'(X)$ be the graph defined as above. We have $G'(X)$ is 3-colorable.*

Proof of Theorem 4: Assume G can be partitioned into k admissible sets, say $V(G) = \cup_{i=1}^k X_i$, where $X_i = X_i^1 \cup X_i^2 \cup X_i^3$. For each $1 \leq i \leq k$ and $1 \leq j \leq 3$, let $Y_i^j = \Gamma(X_i^j)$. From the definition of an admissible set and G being triangle-free, we have Y_i^j is an independent set for all $1 \leq i \leq k$ and $1 \leq j \leq 3$.

By Lemma 9, $G'(X_i)$ is 3-colorable for all $1 \leq i \leq k$. Let c_i be a 3-coloring of $G'(X_i)$ with the color set $\{s_i^1, s_i^2, s_i^3\}$. Here all the colors s_i^j 's are pairwise distinct. We use $\mathcal{P}(S)$ to denote the set of all subsets of S . We define $f_i : V(G) \rightarrow \mathcal{P}(\{s_i^1, s_i^2, s_i^3\})$ satisfying

$$f_i(v) = \begin{cases} \{c_i(v)\} & \text{if } v \in V(G) - (\cup_{j=1}^3 X_i^j \cup \cup_{j=1}^3 Y_i^j), \\ \{c_i(y_i^j)\} & \text{if } v \in Y_i^j, \\ \{s_i^1, s_i^2, s_i^3\} - c_i(y_i^j) & \text{if } v \in X_i^j. \end{cases}$$

Note that for a fixed $1 \leq i \leq k$, y_i^j denotes the vertex of $G'(X_i)$ obtained from contracting Y_i^j for $1 \leq j \leq 3$. Observe that each vertex in X_i^j receives two colors from f_i and every other vertex receives one color. It is clear that any two adjacent vertices receive disjoint colors. Let $\sigma : V(G) \rightarrow \mathcal{P}(\cup_{i=1}^k \{s_i^1, s_i^2, s_i^3\})$ be a mapping defined as $\sigma(v) = \cup_{i=1}^k f_i(v)$. Now σ is a $(k+1)$ -fold coloring of G such that each color is drawn from a palette of $3k$ colors. Thus we have $\chi_f(G) \leq \frac{3k}{k+1} = 3 - \frac{3}{k+1}$.

We completed the proof of theorem 4. \square

Before we prove Lemma 9, we first prove a lemma on coloring the graph obtained by splitting the hub of an odd wheel. Let $\{x_0, x_1, \dots, x_{2k}\}$ be the set

of vertices of an odd cycle C_{2k+1} in a circular order. Let $Y = \{y_1, y_2, y_3\}$. We construct a graph H as follows:

1. $V(H) = V(C) \cup Y$.
2. $E(C) \subset E(H)$.
3. Each x_i is adjacent to exactly one element of Y .
4. $y_1y_2, y_2y_3, y_1y_3 \in E(H)$.
5. H can have at most one vertex (of y_1, y_2 , and y_3) with degree 2.

The graph H can be viewed as splitting the hub of the odd wheel into 3 new hubs where each spoke has to choose one new hub to connect, then connect all new hubs. One special case is that one of the new hubs has no neighbor in C .

Lemma 10 *A graph H constructed as described above is 3-colorable.*

Proof: Without loss of generality, we assume $d_H(y_1) \geq 3$ and $d_H(y_2) \geq 3$. We construct a proper 3-coloring c of H as follows. First, let $c(y_1) = 1$, $c(y_2) = 2$, and $c(y_3) = 3$.

Again, without loss of generality, we assume $(x_0, y_1) \in E(H)$. The neighbors of y_1 divide $V(C)$ into several intervals. The vertices in each interval are either connecting to y_2 or y_3 but not connected to y_1 . As v goes through each interval counter-clockwisely, we list the neighbor of v (in Y) and get a sequence consisting of $y_2 y_3$. Then we delete the repetitions of y_2y_2 and y_3y_3 in the sequence. There are 4 types of intervals based on the result sequence:

Type I: $y_2, y_3, y_2, y_3, \dots, y_3, y_2$.

Type II: $y_2, y_3, y_2, y_3, \dots, y_2, y_3$.

Type III: $y_3, y_2, y_3, y_2, \dots, y_3, y_2$.

Type IV: $y_3, y_2, y_3, y_2, \dots, y_2, y_3$.

If y_3 is a vertex of degree two in H , then the interval I has only one type, which is degenerated into y_2 .

Given an interval I , let $u(I)$ (or $v(I)$) be the common neighbor of y_1 and the left (or right) end of I respectively. We color $u(I)$ and $v(I)$ first and then try to extend it as a proper coloring of I . Sometimes we succeed while sometimes we fail. We ask a question whether we can always get a proper coloring. The answer depends only on the type of I and the coloring combination of $u(I)$ and $v(I)$. In Table 1, the column is classified by the coloring combination of $u(I)$ and $v(I)$, while the row is classified by the types of I . Here “yes” means the coloring process always succeeds, while “no” means it sometimes fails.

An ending vertex w of an interval I is called a *free* end if w 's two neighbors outside I receiving the same color. Observe that in all yes entries, there exist at least one free end. Note that each vertex on I has degree 3. We can always color the vertices of I greedily starting from the end not equaling w . Since w is a free end, there is no difficulty to color w at the end.

Now we put them together. We color the neighbors of y_1 one by one counter-clockwise starting from x_0 according to the following rules:

	(2, 2)	(2, 3)	(3, 2)	(3, 3)
Type I	Yes	Yes	Yes	No
Type II	Yes	Yes	No	Yes
Type III	Yes	No	Yes	Yes
Type IV	No	Yes	Yes	Yes

Table 1: Can a coloring be extended to I properly?

1. When we meet an interval I of type II or III, we keep the colors of $u(I)$ and $v(I)$ the same.
2. When we meet an interval I of type I or IV, we keep the colors of $u(I)$ and $v(I)$ different.

There are two possibilities. If the last interval obeys the rules, then by Table 1, we can extend the partial coloring into a proper 3-coloring of H . If the last interval does not obey the rules, then we swap the colors 2 and 3 of the neighbors of y_1 . By Table 1, the new partial coloring can be extended into a proper 3-coloring of H . We completed the proof. \square

A maximal 2-connected subgraph B of a graph is called a *block* of G . A *Gallai tree* is a connected graph in which all blocks are either complete graphs or odd cycles. A *Gallai forest* is a graph all of whose components are Gallai trees. A *k-Gallai tree (forest)* is a Gallai tree (forest) such that the degree of all vertices are at most $k - 1$. A *k-critical graph* is a graph G whose chromatic number is k and the chromatic number of any proper subgraph is strictly less than k . Gallai showed the following Lemma.

Lemma 11 (Gallai [4]) *If G is a k -critical graph, then the subgraph of G induced on the vertices of degree $k - 1$ is a k -Gallai forest.*

Now, we are ready to prove Lemma 9.

Proof of Lemma 9: Write $G' = G'(X)$ for short. Note that the only possible vertices of degree greater than 3 in G' are y_1 , y_2 , and y_3 . We can color y_1 , y_2 , and y_3 by 1, 2, and 3, respectively. Since the remaining vertices have degree at most 3, we can color G' properly with 4 colors greedily, i.e., $\chi(G') \leq 4$.

Suppose that G' is not 3-colorable. Let H be a 4-critical subgraph of G' . We have $d_H(v) = 3$ for all $v \in H$ except for possible y_1 , y_2 , and y_3 . Let T be the subgraph induced by all $v \in H$ such that $d_H(v) = 3$; then T is not empty since $|T| \geq |H| - 3 \geq 1$. By Lemma 11 the subgraph of H induced by T is a 4-Gallai forest. We know T may contain one or more vertices in $\{y_1, y_2, y_3\}$. Let $T' = T \setminus \{y_1, y_2, y_3\} = V(H) \setminus \{y_1, y_2, y_3\}$. Observe that any induced subgraph of a 4-Gallai forest is still a 4-Gallai forest and so the subgraph of H induced by T' is also a 4-Gallai forest.

Recall the definition of an admissible set. If $u \in X_i$ and $v \in X_j$ for some i and j satisfying $1 \leq i \neq j \leq 3$, then $v \notin N_G^1(u) \cup N_G^2(u) \cup N_G^4(u)$, which implies that any vertex x in T' can have at most one neighbor in $\{y_1, y_2, y_3\}$. Note $d_H(x) = 3$. We have $d_{T'}(x) \geq 2$.

Let B be a leaf block in the Gallai-forest T' ; then B is a complete graph or an odd cycle by the definition of the Gallai-forest. Observe B can not be a single vertex or K_2 since every vertex in B has at least two neighbors in T' . As G is triangle-free, then B must be an odd cycle C_{2r+1} with $r \geq 2$.

Case (a): $|N_H(B) \cap \{y_1, y_2, y_3\}| \geq 2$. Since H is 4-critical, $H \setminus B$ is 3-colorable. Let c be a proper 3-coloring of $H \setminus B$. Since $H \setminus B$ contains a triangle $y_1 y_2 y_3$, we have y_1, y_2 , and y_3 receive different colors. We have

$$|N_H(B) \cap \{y_1, y_2, y_3\}| \geq 2.$$

By Lemma 10, we can extend the coloring c to all vertices on B as well. Thus H is 3-colorable. Contradiction.

Case (b): $|N_H(B) \cap \{y_1, y_2, y_3\}| = 1$. Since B is a leaf block, there is at most one vertex, say v_0 , which is connected to another block in T' . List the vertices of B in a circular order as $v_0, v_1, v_2, \dots, v_{2r}$. All v_1, v_2, \dots, v_{2r} connect to one y_i , say y_1 , which implies that for $1 \leq i \leq 2r$, there exists a vertex $x_i \in X_1$ and a vertex $w_i \in Y_1$ so that $v_i - w_i - x_i$ form a path of length 2. Since G is triangle-free, we have $w_i \neq w_{i+1}$ for all $i \in \{1, \dots, 2r-1\}$. Note that $x_i - w_i - v_i - v_{i+1} - w_{i+1} - x_{i+1}$ forms a path of length 5 unless $x_i = x_{i+1}$.

Recall the admissible conditions: if $\{u, v\} \subset X_i$ for some $i \in \{1, 2, 3\}$, then $v \notin N_G^1(u) \cup N_G^3(u) \cup N_G^5(u)$. We must have $x_1 = x_2 = \dots = x_{2r}$. Denote this common vertex by x . Now have

$$|N_G^2(x) \cap B| \geq 2r.$$

Note $|N_G^2(x)| \leq 6$. We have $2r \leq 6$. The possible values for r are 2 and 3. If $r = 2$, then B is a 5-cycle. Since B is in T' , we have $B \cap N_G^1(x) = \emptyset$; this is a contradiction to Lemma 7. If $r = 3$, then B is a 7-cycle; this is a contradiction to Lemma 8.

The proof of Lemma 9 is finished. \square

4 Partition into 42 admissible sets

Theorem 5 *Let G be a triangle-free graph with maximum degree at most 3. If G is 2-connected and $\text{girth}(G) \leq 6$, then G can be partitioned into at most 42 admissible sets.*

Proof of Theorem 5: We will define a proper coloring $c: V(G) \rightarrow \{1, 2, \dots, 126\}$ such that for $1 \leq i \leq 42$, the i -th admissible set is $c^{-1}(\{3i-2, 3i-1, 3i\})$. We refer to $\{3i-2, 3i-1, 3i\}$ as a color block for all $i \in \{1, \dots, 42\}$. Since $4 \leq \text{girth}(G) \leq 6$, there is a cycle C of length 4, 5, or 6. Let v_{n-1} and v_n be a pair of adjacent vertices of C . We assume that $G \setminus \{v_{n-1}, v_n\}$ is connected. (If not, we start the greedy algorithm below from each of the components of

$G \setminus \{v_{n-1}, v_n\}$ separately.) We can find a vertex v_1 other than v_{n-1} and v_n such that $G \setminus v_1$ is connected. Inductively, for each $i \in \{2, \dots, n-2\}$, we can find a vertex v_j other than v_{n-1} and v_n such that $G \setminus \{v_1, \dots, v_{j-1}\}$ is connected. Therefore, we get an order of vertices $v_1, v_2, \dots, v_{n-1}, v_n$ such that for $j = 1, 2, \dots, n-2$, the induced graph on v_j, \dots, v_n is connected.

We color the vertices greedily. Assume we have colored v_1, v_2, \dots, v_j . For v_{j+1} , choose a color h satisfying the following:

1. For each $u \in N_G^1(v_{j+1}) \cap \{v_1, v_2, \dots, v_j\}$, we have h is not in the same block of $c(u)$.
2. For each $u \in (N_G^3(v_{j+1}) \cup N_G^5(v_{j+1})) \cap \{v_1, v_2, \dots, v_j\}$, we have $h \neq c(u)$.
3. For each $u \in (N_G^2(v_{j+1}) \cup N_G^4(v_{j+1})) \cap \{v_1, v_2, \dots, v_j\}$, we have h could equal $c(u)$ but not equal the other two colors in the color block of $c(u)$.

For $j \leq n-2$, there is at least one vertex in $N_G^1(v_{j+1})$ and one vertex in $N_G^2(v_{j+1})$ still uncolored. Thus $|N_G^1(v_{j+1}) \cap \{v_1, v_2, \dots, v_j\}| \leq 2$, $|N_G^2(v_{j+1}) \cap \{v_1, v_2, \dots, v_j\}| \leq 5$, $|N_G^3(v_{j+1})| \leq 12$, $|N_G^4(v_{j+1})| \leq 24$, and $|N_G^5(v_{j+1})| \leq 48$. Since

$$3 \times 2 + 2 \times (5 + 24) + (12 + 48) = 124 < 126,$$

it is always possible to color the vertex v_{j+1} properly.

It remains to color v_{n-1} and v_n properly. Note both v_{n-1} and v_n are on the cycle C . Let us count color redundancy according to the type of the cycle C .

Case C_4 : For any vertex v on C_4 , there are two vertices in $N_G^1(v) \cap N_G^3(v)$. We have $|N_G^2(v)| \leq 5$ and $|N_G^4(v)| \leq 23$. Thus the number of colors forbidden to be assigned to v is at most

$$3 \times 3 + 2 \times (5 + 23) + (12 + 48) - 2 = 123 < 126.$$

Case C_5 : For any vertex v on C_5 , there are two vertices in $N_G^1(v) \cap N_G^4(v)$. We also have $|N_G^5(v)| \leq 47$. The number of colors forbidden to be assigned to v is at most

$$3 \times 3 + 2 \times (6 + 24) + (12 + 47) - 2 \times 2 = 124 < 126.$$

Case C_6 : For any vertex v on C_6 , there are two vertices in $N_G^1(v) \cap N_G^5(v)$ and two vertices in $N_G^2(v) \cap N_G^4(v)$. We have $|N_G^3(v)| \leq 11$. Thus the number of colors forbidden to be assigned to v is at most

$$3 \times 3 + 2 \times (6 + 24) + (11 + 48) - 2 - 2 \times 2 = 122 < 126.$$

In each subcase, we can find a color for v_{n-1} and v_n .

The 42 admissible sets can be obtained from the coloring c as follows. For $1 \leq j \leq 42$, the j -th admissible set has the following partition

$$c^{-1}(3i-2) \cup c^{-1}(3i-1) \cup c^{-1}(3i).$$

Those are admissible sets by the construction of the coloring c . \square

Proof of Theorem 1: Suppose that there exists a graph G which is triangle-free, $\Delta \leq 3$, and $\chi_f(G) > 3 - \frac{3}{43}$. Without loss of generality, we can assume G has the smallest number of edges among all such graphs. Thus G is 2-connected and fractionally-critical. If $\text{girth}(G) \geq 7$, Hatami and Zhu [7] showed $\chi_f(G) \leq 2.78571 \leq 3 - \frac{3}{43}$. Contradiction!

If $\text{girth}(G) \leq 6$, Theorem 5 states that G can be partitioned into 42 admissible sets. By Theorem 4, we have $\chi_f(G) \leq 3 - \frac{3}{k+1} = 3 - \frac{3}{43}$. Contradiction! \square

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